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LOCAL AND GLOBAL BIFURCATION PHENOMENA IN PLANE-STRAIN FINITE ELASTICITY

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Abstract—Bifurcation, global non-uniqueness and stability of solutions to the plane-strain problem of an incompressible isotropic elastic material subject to in-plane dead-load tractions are considered. In particular, for loading in equibiaxial tension, bifurcation from a configuration in which the in-plane principal stretches are equal is shown to occur at a certain critical value of the tension (which depends on the form of strain-energy function). Results concerning the global invertibility of the elastic stress-deformation relations are obtained and then used to derive an equation governing the deformation paths branching from this critical value. The stability of each branch is also examined. The analysis is carried through for a general form of strain-energy function and the results are then illustrated for a particular class of strain-energy functions.

1. INTRODUCTION

Non-uniqueness of solution to the problem of a cube of incompressible isotropic elastic material subject to three equal pairs of equal and opposite dead-load tractions acting normally to its faces has been examined by Rivlin [8, 9] in respect of the neo-Hookean form of strain-energy function. Rivlin also determined the stability of the solutions; Hill [3] pointed out some deficiencies in Rivlin's [8] stability analysis which were subsequently corrected [9]. Somewhat different aspects of the same problem have been considered recently by Ball and Schaeffer [1]; in particular, they have examined the branching of solutions from a configuration of pure dilatation as the applied load increases from zero and the stability of the branches. Their work made extensive use of the Mooney form of strain-energy function and they used singularity theory to examine the local aspects of bifurcation. The closely-related problem of non-uniqueness and stability of a cube subject to two equal pairs of tractions differing from the third pair has been discussed by Sawyers [10] with attention restricted to the neo-Hookean strain-energy function.

In the present paper we consider a slightly different problem, namely the *plane-strain* bifurcation, non-uniqueness and stability of a body subject to in-plane equibiaxial dead-load tractions. The material is taken to be incompressible and isotropic, but no other restriction is placed on the form of elastic strain-energy function.

We begin by giving a brief general account of the governing equations and their application to the study of bifurcation and stability in order to place the problem under consideration within a broader framework.

2. THE BASIC EQUATIONS

Consider a material body which occupies the region B_0 in some reference configuration, and suppose that the *deformation* $\chi : B_0 \rightarrow B$ defines the region B occupied by the body in the current configuration. Let $\mathbf{X}$ denote a typical point of B_0 and also its position vector relative to an arbitrary choice of origin. Similarly, let $\mathbf{x}$ denote the position vector of $\mathbf{X}$ in B, so that

$$\mathbf{x} = \chi(\mathbf{X}) \qquad \mathbf{X} \in B_0. \tag{1}$$

The deformation gradient tensor $\underset{\sim}{A}$ is defined by

$$\underset{\sim}{A} = \text{Grad}\, \chi, \tag{2}$$

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where Grad denotes the gradient operator with respect to $\mathbf{X}$. Use will be made of the polar decomposition

$$\mathbf{A} = \mathbf{R}\mathbf{U}, \tag{3}$$

where $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$ is the positive definite, symmetric *right stretch tensor*. In this paper attention is restricted to incompressible materials so that we have the constraint

$$\det \mathbf{A} = \det \mathbf{U} = 1. \tag{4}$$

For an (incompressible) elastic material the *nominal stress tensor* $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1}, \tag{5}$$

where p is the arbitrary hydrostatic pressure (Lagrange multiplier) arising from the constraint eqn (4) and $W(\mathbf{A})$ is the elastic strain energy per unit volume.

The strain-energy is *objective*, i.e. unaffected by a superposed rigid-body rotation after deformation, and it therefore follows from eqn (3) that W depends on $\mathbf{A}$ only through $\mathbf{U}$. Thus

$$W(\mathbf{A}) \equiv W(\mathbf{U}). \tag{6}$$

This leads to an alternative and very useful form of stress-deformation relation, namely

$$\mathbf{T}^{(1)} = \frac{\partial W}{\partial \mathbf{U}} - p\mathbf{U}^{-1}, \tag{7}$$

where $\mathbf{T}^{(1)}$ is the symmetric Biot stress tensor†.

For an *isotropic* material W depends on $\mathbf{U}$ only through its principal values λ_1, λ_2, λ_3, the *principal stretches*. We write this dependence as $W(\lambda_1, \lambda_2, \lambda_3)$ and note the symmetries

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_3, \lambda_1, \lambda_2), \tag{8}$$

with

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{9}$$

following from eqn (4).

If $t_i^{(1)}$ $(i = 1, 2, 3)$ denote the principal components of $\mathbf{T}^{(1)}$ then, for an isotropic material, eqn (7) gives

$$t_i^{(1)} = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1} \qquad i = 1, 2, 3. \tag{10}$$

Also, for an isotropic material, we have the connection

$$\frac{\partial W}{\partial \mathbf{A}} = \frac{\partial W}{\partial \mathbf{U}}\mathbf{R}^T$$

† The notation $\mathbf{T}^{(1)}$ is used because $\mathbf{T}^{(1)}$ is conjugate to $\mathbf{U}-\mathbf{I}$, corresponding to $m = 1$ in the class $(\mathbf{U}^m - \mathbf{I})/m$ of strain tensors. For a full account of conjugate stress and strain tensors see [7].

(the proof of this is straightforward), and hence, by eqns (3), (5) and (7),

$$\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T. \tag{11}$$

In terms of the principal components t_i ($i = 1, 2, 3$) of the Cauchy stress tensor $\mathbf{T} = \mathbf{A}\mathbf{S}$ we have

$$t_i = \lambda_i t_i^{(1)} = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \qquad i = 1, 2, 3. \tag{12}$$

Finally in this section we record that in the absence of body forces the equilibrium equation is simply

$$\text{Div}\, \mathbf{S} = \mathbf{0} \quad \text{in} \quad B_0, \tag{13}$$

where Div is the divergence operator with respect to **X**. Typical boundary conditions involve the specification of **x** and the traction $\mathbf{S}^T\mathbf{N}$ on complementary parts of the boundary ∂B_0 of B_0, **N** being the unit outward normal to ∂B_0.

3. THE INCREMENTAL EQUATIONS

Let $(\dot{\chi})$ denote an increment in χ, i.e. a small deformation from the current configuration. Then, on taking the increment of eqn (5), we obtain the incremental constitutive law

$$\dot{\mathbf{S}} = \mathcal{A}^1\dot{\mathbf{A}} - \dot{p}\mathbf{A}^{-1} + p\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}, \tag{14}$$

where $\dot{\mathbf{A}} = \text{Grad}\, \dot{\chi}$, $\dot{\mathbf{S}}$ and $\dot{p}$ denote increments in $\mathbf{S}$ and p and

$$\mathcal{A}^1 = \frac{\partial^2 W}{\partial \mathbf{A}^2}$$

is the (fourth-order) tensor of first-order elastic moduli associated with the conjugate pair ($\mathbf{S}$, $\mathbf{A}$). The right-hand side of eqn (14) is correct to the first order in incremental quantities.

It is convenient for our purposes to choose the reference configuration to coincide with the current configuration, in which case eqn (14) becomes

$$\dot{\mathbf{S}}_0 = \mathcal{A}_0^1\dot{\mathbf{A}}_0 - \dot{p}\mathbf{I} + p\dot{\mathbf{A}}_0, \tag{15}$$

where $\mathbf{I}$ is the (second-order) identity tensor and the subscript zero signifies evaluation in the current configuration. The tensor $\mathcal{A}_0^1$ is called the tensor of first-order *instantaneous moduli* associated with ($\mathbf{S}$, $\mathbf{A}$), while

$$\dot{\mathbf{A}}_0 = \dot{\mathbf{A}}\mathbf{A}^{-1} = \text{grad}\, \mathbf{v}, \tag{16}$$

where $\mathbf{v}(\mathbf{x})$ is identified with $\dot{\chi}(\mathbf{X})$ through (1) and grad denotes the gradient operator with respect to **x**.

Correct to the first order in incremental quantities the incremental form of the constraint eqn (4) is

$$\text{tr}(\dot{\mathbf{A}}_0) \equiv \text{div}\, \mathbf{v} = 0, \tag{17}$$

div representing the divergence with respect to **x**.

For an isotropic material we shall require the components of $\mathscr{A}_0^1$ referred to the underlying principal axes (i.e. the principal axes of $\underset{\sim}{T}$). These are given by

$$\left.\begin{aligned} \mathscr{A}^1_{0iijj} &= \lambda_i\lambda_j \frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \\ \mathscr{A}^1_{0ijij} &= \begin{cases} (t_i - t_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2) & \lambda_i \neq \lambda_j, \quad i \neq j, \\ \dfrac{1}{2}\left(\mathscr{A}^1_{0iiii} - \mathscr{A}^1_{0iijj} + \lambda_i \dfrac{\partial W}{\partial \lambda_i}\right) & \lambda_i = \lambda_j, \quad i \neq j, \end{cases} \\ \mathscr{A}^1_{0ijji} &= \mathscr{A}^1_{0jiij} = \mathscr{A}^1_{0ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} \qquad i \neq j. \end{aligned}\right\} \tag{18}$$

Derivations of these expressions are contained in [5] and fuller details in [7].

Referred to the current configuration the incremental counterpart of eqn (13) is

$$\operatorname{div} \dot{\underset{\sim}{S}}_0 = \mathbf{0} \quad \text{in} \quad B, \tag{19}$$

and the boundary conditions may specify, for example, $\mathbf{v}$ and the incremental traction $\dot{\underset{\sim}{S}}_0^T\mathbf{n}$ on complementary parts of ∂B, where $\mathbf{n}$ is the unit outward normal to ∂B.

4. INCREMENTAL UNIQUENESS AND STABILITY

Suppose that χ is a solution of the underlying problem and that the incremental problem has homogeneous boundary conditions. It follows that $(\dot{\chi}) = \mathbf{0}$ is a solution of the incremental problem. If $(\dot{\chi}) = \mathbf{v} \not\equiv \mathbf{0}$ is also a solution then, by use of eqns (16), (19) and the divergence theorem, we obtain

$$\int_B \operatorname{tr}(\dot{\underset{\sim}{S}}_0\dot{\underset{\sim}{A}}_0)\,\mathrm{d}V = \int_{\partial B} \mathbf{v}\cdot(\dot{\underset{\sim}{S}}_0^T\mathbf{n})\,\mathrm{d}S = 0,$$

where $\mathrm{d}V$, $\mathrm{d}S$ denote volume and surface elements in B and ∂B respectively. Hence, incremental uniqueness (of both the problem with homogeneous data and that with inhomogeneous data) is guaranteed if

$$\int_B \operatorname{tr}(\dot{\underset{\sim}{S}}_0\dot{\underset{\sim}{A}}_0)\,\mathrm{d}V \neq 0$$

for all (twice continuously differentiable) $\mathbf{v}$ satisfying $\mathbf{v} = \mathbf{0}$ on the appropriate part of ∂B and eqn (17) in B. We refer to such $\mathbf{v}$ as *admissible*.

The configuration χ is said to be *unstable* if the above functional is negative for *some* admissible $\mathbf{v}$ and *stable* if

$$\int_B \operatorname{tr}(\dot{\underset{\sim}{S}}_0\dot{\underset{\sim}{A}}_0)\,\mathrm{d}V \geq 0 \tag{20}$$

for *all* admissible $\mathbf{v}$. If strict equality holds in eqn (20) for some admissible $\mathbf{v}$ with grad $\mathbf{v}$ $\not\equiv \underset{\sim}{0}$ then χ is said to be *neutrally stable*. However, this description is valid only to the second order in incremental quantities and a neutrally stable configuration may, in fact, be unstable when assessed to higher order in $\dot{\underset{\sim}{A}}_0$ (the left-hand side of eqn (20), which represents twice the increase in total energy of the body under dead-loading conditions due to the increment $(\dot{\chi})$, must then include such higher-order terms). For full discussion of this we refer to [2, 4 and 7].

When eqn (20) is strengthened to

$$\int_B \mathrm{tr}(\dot{\mathbf{S}}_0\dot{\mathbf{A}}_0)\,\mathrm{d}V > 0 \tag{21}$$

for all admissible v with $\dot{\mathbf{A}}_0 \neq \mathbf{0}$, the current configuration is stable and the incremental problem has a unique solution. Thus eqn (21) excludes bifurcation and is therefore referred to as the *exclusion condition* [4].

Suppose henceforth that the material properties and the deformation χ are *homogeneous*. Suppose further that the body is subject to *all-round dead load*. Then eqn (21) is equivalent to

$$\mathrm{tr}(\dot{\mathbf{S}}_0\dot{\mathbf{A}}_0) > 0 \tag{22}$$

for all $\dot{\mathbf{A}}_0$ satisfying eqn (17), or, from eqn (15),

$$\mathrm{tr}\{(\mathcal{A}_0^1\dot{\mathbf{A}}_0)\dot{\mathbf{A}}_0 + p\dot{\mathbf{A}}_0^2\} > 0, \tag{23}$$

independently of the geometry of the body. Along a deformation path on which eqn (23) holds the exclusion condition first fails where

$$\mathrm{tr}\{(\mathcal{A}_0^1\dot{\mathbf{A}}_0)\dot{\mathbf{A}}_0 + p\dot{\mathbf{A}}_0^2\} \geq 0 \tag{24}$$

for all $\dot{\mathbf{A}}_0$ satisfying eqn (17) with equality holding for some $\dot{\mathbf{A}}_0 \neq \mathbf{0}$.

Extremizing eqn (24) shows that in a neutrally stable configuration

$$\dot{\mathbf{S}}_0 \equiv \mathcal{A}_0^1\dot{\mathbf{A}}_0 + p\dot{\mathbf{A}}_0 - \dot{p}\mathbf{I} = \mathbf{0}, \tag{25}$$

where $\dot{p}$ is the Lagrange multiplier introduced in respect of the constraint eqn (17), i.e. the nominal stress is stationary with respect to incremental deformations which minimize the quadratic form eqn (24).

For an isotropic material use of eqns (17), (18), (12) and a little algebra shows that the exclusion condition eqn (23) may be arranged in the form

$$\begin{aligned}
&\left\{\left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_3\frac{\partial}{\partial\lambda_3}\right)^2 W - t_1 - t_3\right\} v_{11}^2 \\
&+ \left\{\left(\lambda_2\frac{\partial}{\partial\lambda_2} - \lambda_3\frac{\partial}{\partial\lambda_3}\right)^2 W - t_2 - t_3\right\} v_{22}^2 \\
&+ \left\{\left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_3\frac{\partial}{\partial\lambda_3}\right)^2 W + \left(\lambda_2\frac{\partial}{\partial\lambda_2} - \lambda_3\frac{\partial}{\partial\lambda_3}\right)^2 W \right. \\
&\left. - \left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_2\frac{\partial}{\partial\lambda_2}\right)^2 W - 2t_3\right\} v_{11}v_{22} \\
&+ \frac{1}{2}\sum_{i,j\neq i\in\{1,2,3\}}\left\{\left(\frac{t_i - t_j}{\lambda_i^2 - \lambda_j^2}\right)(\lambda_i^2 v_{ji}^2 + \lambda_j^2 v_{ij}^2)\right. \\
&\left. + 2\left(\frac{t_i\lambda_j^2 - t_j\lambda_i^2}{\lambda_i^2 - \lambda_j^2}\right) v_{ij}v_{ji}\right\} > 0,
\end{aligned} \tag{26}$$

where v_{ij} are the components of grad v on the principal axes of $\mathbf{T}$. It is an easy matter to obtain necessary and sufficient conditions for eqn (26) to hold for all grad v $\neq \mathbf{0}$;

for the plane strain specialization of eqn (26) such conditions are given in the following section.

For an isotropic material, neutrally stable configurations can be found either by use of eqn (25) or by direct extremization of eqn (26). Henceforth we restrict attention to the plane strain problem for isotropic materials.

5. PLANE STRAIN BIFURCATION CRITERIA

Suppose that the stretch λ_3 is fixed and normal to the plane in question and that the material is subject to a pure homogeneous strain with in-plane principal stretches λ_1 and λ_2. Let the principal axes of the strain define Cartesian axes with coordinates x_1, x_2, x_3. We restrict the incremental deformation $\mathbf{v}$ so that $v_3 = 0$ and v_1, v_2 are independent of x_3. Equation (17) reduces to

$$v_{11} + v_{22} = 0 \tag{27}$$

and the exclusion condition (26) to

$$\left\{\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \lambda_2 \frac{\partial}{\partial\lambda_2}\right)^2 W - t_1 - t_2\right\} v_{11}^2 + \left(\frac{t_1 - t_2}{\lambda_1^2 - \lambda_2^2}\right)(\lambda_1^2 v_{21}^2 + \lambda_2^2 v_{12}^2) + 2\left(\frac{t_1\lambda_2^2 - t_2\lambda_1^2}{\lambda_1^2 - \lambda_2^2}\right) v_{12}v_{21} > 0, \tag{28}$$

where $v_{ij} = \partial v_i/\partial x_j$.

Necessary and sufficient conditions for eqn (28) to hold are, with the help of eqn (12), easily seen to be

$$\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \lambda_2 \frac{\partial}{\partial\lambda_2}\right)^2 W - t_1 - t_2 > 0, \tag{29}$$

$$t_1^{(1)} + t_2^{(1)} > 0, \tag{30}$$

$$(t_1^{(1)} - t_2^{(1)})/(\lambda_1 - \lambda_2) > 0, \tag{31}$$

jointly, and, by a limiting process, it can also be seen that eqn (31) is equivalent to eqn (29) when $\lambda_1 = \lambda_2$. Corresponding inequalities for compressible materials are well known [4] but for incompressible materials (29)–(31) are apparently new although eqns (30) and (31) are identical in form to their compressible counterparts.

Neutrally stable configurations are defined by values of λ_1 and λ_2 for which one or more of eqns (29)–(31) just fails. Such values define bifurcation points in the (λ_1, λ_2)-plane. The bifurcation criterion

$$\left(\lambda_1 \frac{\partial}{\partial\lambda_1} - \lambda_2 \frac{\partial}{\partial\lambda_2}\right)^2 W - t_1 - t_2 = 0 \tag{32}$$

is associated with a pure shear mode of incremental deformation ($v_{22} = -v_{11}$) coaxial with the underlying pure strain, while

$$t_1^{(1)} + t_2^{(1)} = 0 \tag{33}$$

and

$$(t_1^{(1)} - t_2^{(1)})/(\lambda_1 - \lambda_2) = 0 \tag{34}$$

permit in-plane shearing modes such that $v_{21}/v_{12} = -\lambda_2/\lambda_1$ and λ_2/λ_1 respectively.

Equations (32)–(34) may be derived directly from eqn (25), with eqn (32) requiring elimination of $\dot{p}$.

The (nominal) stress-deformation relation is clearly *locally* invertible except in configurations such as those just described and, more generally, in configurations where eqn (25) holds for $\dot{\mathbf{A}}_0 \neq \mathbf{0}$. Given $\dot{\mathbf{S}}_0$ in a configuration where the exclusion condition holds, $\dot{\mathbf{A}}_0$ is uniquely defined by eqn (15), $\dot{p}$ being fixed by the constraint $\mathrm{tr}(\dot{\mathbf{A}}_0) = 0$.

6. APPLICATION TO THE CASE OF EQUIBIAXIAL TENSION

At this point it is convenient to introduce the auxiliary variable λ defined by

$$\lambda_1 = \lambda\lambda_3^{-1/2}, \qquad \lambda_2 = \lambda^{-1}\lambda_3^{-1/2} \tag{35}$$

together with the notation

$$\hat{W}(\lambda, \lambda_3) = W(\lambda\lambda_3^{-1/2}, \lambda^{-1}\lambda_3^{-1/2}, \lambda_3). \tag{36}$$

From the symmetry of eqn (8) we deduce that

$$\hat{W}(\lambda^{-1}, \lambda_3) = \hat{W}(\lambda, \lambda_3), \tag{37}$$

a result which will be required in Section 7.

From eqns (12) and (36) we obtain

$$t_1 - t_2 = \lambda \frac{\partial \hat{W}}{\partial \lambda} \tag{38}$$

and

$$\left(\lambda_1 \frac{\partial}{\partial \lambda_1} - \lambda_2 \frac{\partial}{\partial \lambda_2}\right)^2 W - t_1 - t_2 = \lambda^2 \frac{\partial^2 \hat{W}}{\partial \lambda^2} - 2t_2, \tag{39}$$

the latter expression occurring in eqn (32).

Of particular interest is the deformation for which $\lambda_1 = \lambda_2$, i.e. $\lambda = 1$, with $t_1 = t_2$ correspondingly, when the exclusion condition (28) reduces to

$$\left\{\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3) - 2t_1\right\}\{v_{11}^2 + \tfrac{1}{4}(v_{12} + v_{21})^2\} + \tfrac{1}{4}t_1(v_{12} - v_{21})^2 > 0. \tag{40}$$

Necessary and sufficient conditions for eqn (40) to hold for all v_{ij} with $v_{ij} \neq 0$ for some pair i, j are simply

$$0 < t_1 < \frac{1}{2}\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3). \tag{41}$$

On a path of equibiaxial tensile loading with $\lambda = 1$ bifurcation becomes possible when t_1 reaches the value

$$t_1 = \frac{1}{2}\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3) \tag{42}$$

and for larger values of t_1 the deformation $\lambda = 1$ is clearly unstable. From the results of Section 5 we deduce that a symmetry-breaking mode of deformation appears at the value of t_1 defined by eqn (42). This deformation comprises a pure shear $v_{22} = -v_{11}$ coaxial with the chosen Cartesian axes and a shearing mode with $v_{12} = v_{21}$. The combination of these modes represents a pure shear deformation whose axes have orientation dependent on v_{11}/v_{12} (which is arbitrary). This arbitrariness is a consequence of the fact that the orientation of the in-plane principal axes of $\underset{\sim}{T}^{(1)}$ is arbitrary since $t_1^{(1)} = t_2^{(1)}$ in the considered configuration. This latter point also arises in connection with the *global* inversion of eqn (5), as we see in the following section.

7. GLOBAL RESULTS

According to eqn (11), for an isotropic material we may decompose the nominal stress as

$$\underset{\sim}{S} = \underset{\sim}{T}^{(1)}\underset{\sim}{R}^T. \tag{43}$$

However, for a given S this polar decomposition, unlike eqn (3), is not unique since $\underset{\sim}{T}^{(1)}$ need not be positive (or negative) definite. When $\underset{\sim}{S}\underset{\sim}{S}^T$ has distinct principal values it is known [6, 7] that, for given $\underset{\sim}{S}$, there are just four distinct polar decompositions of the form eqn (43) and, moreover, only one of these satisfies the stability requirements

$$t_i^{(1)} + t_j^{(1)} > 0 \qquad i \neq j. \tag{44}$$

By contrast, if two principal values of $\underset{\sim}{S}\underset{\sim}{S}^T$ are equal and nonzero, the principal values $t_i^{(1)}$ ($i = 1, 2, 3$) of $\underset{\sim}{T}^{(1)}$ are again uniquely determined under the requirements eqn (44), but the orientation of the principal axes of $\underset{\sim}{T}^{(1)}$ in the plane of the equal values is arbitrary [6, 7]. In particular, this is the case for the equibiaxial problem discussed in Section 6.

Once $\underset{\sim}{T}^{(1)}$, and hence $\underset{\sim}{R}$, has been obtained from eqn (43) with eqn (44) to within the arbitrariness just mentioned, the right stretch tensor $\underset{\sim}{U}$, and hence $\underset{\sim}{A} = \underset{\sim}{R}\underset{\sim}{U}$, is to be found by inverting eqn (7) subject to eqn (4). For the equibiaxial tension problem $t_1^{(1)} = t_2^{(1)}$ and the in-plane principal axes of $\underset{\sim}{T}^{(1)}$, and hence of $\underset{\sim}{U}$, have arbitrary orientation in the (1, 2)-plane. Since λ_3 is fixed it therefore remains to find λ_1 and λ_2, subject to eqn (9), together with p, if required, in terms of $t_1^{(1)}$ from eqn (10).

Elimination of p from eqn (12) gives

$$\lambda_1 t_1^{(1)} - \lambda_1 \frac{\partial W}{\partial \lambda_1} = \lambda_2 t_2^{(1)} - \lambda_2 \frac{\partial W}{\partial \lambda_2} = \lambda_3 t_3^{(1)} - \lambda_3 \frac{\partial W}{\partial \lambda_3}. \tag{45}$$

The first equation in (45) serves to determine λ_1 and λ_2 for fixed λ_3 and prescribed $t_1^{(1)}$ and $t_2^{(1)}$, while the second equation then determines $t_3^{(1)}$.

Now set $t_1^{(1)} = t_2^{(1)}$ and introduce the notation

$$t^{(1)} = \lambda_3^{-1/2} t_1^{(1)}. \tag{46}$$

Clearly, the first equation in (45) is satisfied for all $t^{(1)}$ and λ_3 when $\lambda_1 = \lambda_2$ ($\lambda = 1$ in the notation of eqn (35)). On use of eqns (35), (36) and (46) we see that a solution with $\lambda \neq 1$ is governed by

$$t^{(1)} = \lambda^2 \frac{\partial \hat{W}}{\partial \lambda} / (\lambda^2 - 1) \tag{47}$$

and that in the limit $\lambda \to 1$ this becomes

$$t^{(1)} = t_1 = \frac{1}{2} \frac{\partial \hat{W}}{\partial \lambda^2}(1, \lambda_3) \equiv t_c^{(1)}, \tag{48}$$

wherein the critical value $t_c^{(1)}$ is defined. Recalling eqn (42) we note that this is precisely the critical value at which bifurcation can occur on the fundamental deformation path

$\lambda = 1$ as $t^{(1)}$ increases from zero. Thus, eqn (47) governs the global path of deformation emanating from the point $(t_c^{(1)}, 1)$ in the $(t^{(1)}, \lambda)$-plane.

As we indicated in Section 6 the fundamental path is unstable for values of $t^{(1)}$ greater than $t_c^{(1)}$. In order to assess the stability of the path described by eqn (47) we note that the exclusion condition of eqn (28) simplifies to

$$\left(\lambda^2 \frac{\partial \hat{W}}{\partial \lambda^2} - 2t_2\right) v_{11}^2 + t^{(1)}(\lambda_1 v_{21} - \lambda_2 v_{12})^2/(\lambda_1 + \lambda_2) > 0. \tag{49}$$

However, the left-hand side of eqn (49) vanishes for shearing modes governed by $\lambda_1 v_{21} = \lambda_2 v_{12}$ if $v_{11} = 0$ (recall that eqn (34) holds). It follows that the deformation branch governed by eqn (47) is (neutrally) stable provided

$$0 \leq t^{(1)} \leq \tfrac{1}{2}\lambda^3 \frac{\partial^2 \hat{W}}{\partial \lambda^2}$$

or, equivalently,

$$\lambda^2 \frac{\partial^2 \hat{W}}{\partial \lambda^2} \geq 2\lambda \frac{\partial \hat{W}}{\partial \lambda}/(\lambda^2 - 1) \geq 0. \tag{50}$$

In order to illustrate the consequences of the inequality of eqn (50) we consider the class of strain-energy functions defined by

$$W(\lambda_1, \lambda_2, \lambda_3) = \mu(\lambda_1^m + \lambda_2^m + \lambda_3^m - 3)/m^2,$$

where $\mu > 0$ is the shear modulus.

From eqn (36) we obtain

$$\hat{W}(\lambda, \lambda_3) = \mu(\lambda^m \lambda_3^{-(1/2)m} + \lambda^{-m}\lambda_3^{-(1/2)m} + \lambda_3^m - 3)/m^2 \tag{51}$$

and hence

$$\lambda \frac{\partial \hat{W}}{\partial \lambda} = \mu(\lambda^m - \lambda^{-m})\lambda_3^{-(1/2)m}/m \tag{52}$$

so that the left-hand inequality in eqn (50) can be rearranged as

$$f_m(\lambda) \equiv \{(m - 1)\lambda^{m+1} - (m + 1)\lambda^{m-1} + (m + 1)\lambda^{-m+1} - (m - 1)\lambda^{-m-1}\}/(\lambda - \lambda^{-1}) \geq 0. \tag{53}$$

This is clearly symmetric with respect to interchange of λ and λ^{-1} and reduces to zero in the limit $\lambda \to 1$ for any m.

For $|m| = 1$, $f_m(\lambda) \equiv 0$ while

$$f_m(\lambda) \begin{cases} \geq 0 & \text{for} \quad |m| > 1 \\ \leq 0 & \text{for} \quad |m| < 1 \end{cases}$$

with equality if and only if $\lambda = 1$. It follows that for strain-energy functions with $|m| \geq 1$ the deformation branch is (neutrally) stable, but for those with $|m| < 1$ it is unstable.

The branching of the deformation path is illustrated in Figure 1 where λ is plotted against $t^{(1)}$ for a typical value of $|m|$ in each of the ranges $|m| < 1$ and $|m| > 1$ on the basis of eqn (47) with eqn (52). In each case the curves above and below the line $\lambda = 1$ are obtained one from the other by reversal of the roles of λ and λ^{-1}, i.e. of λ_1

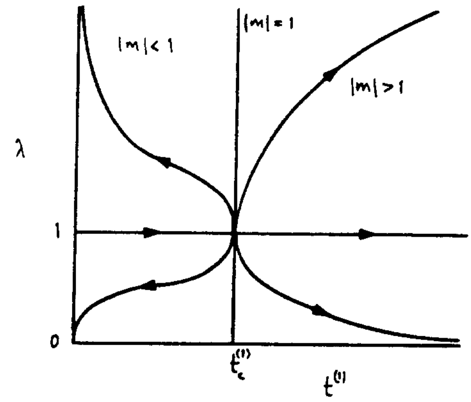

Fig. 1. Bifurcation diagram in the $(t^{(1)}, \lambda)$-plane showing branches emanating from bifurcation point $t^{(1)} = t_c^{(1)}$ on the path $\lambda = 1$ for different values of $|m|$.

and λ_2. Thus, for a body of square cross section, for example, under biaxial deformation the curves represent geometrically equivalent deformations.

It is instructive to examine the behaviour of the branches in the neighbourhood of the bifurcation point and for this purpose we differentiate eqn (47) with respect to λ and evaluate the result for $\lambda = 1$. One differentiation yields

$$\frac{\partial t^{(1)}}{\partial \lambda} = 0 \quad \text{for} \quad \lambda = 1 \tag{54}$$

for arbitrary $\hat{W}$ and λ_3. To obtain this result a limiting process must be used together with $\partial \hat{W}(1, \lambda_3)/\partial\lambda = 0$, which follows from eqn (38), and the connection

$$\frac{\partial^3 \hat{W}}{\partial \lambda^3}(1, \lambda_3) = -3\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3), \tag{55}$$

which is obtained by differentiating eqn (37) three times with respect to λ.

Equation (54) shows that, as indicated in Figure 1, the initial gradient to each branch is vertical irrespective of the form of $\hat{W}$. The initial curvature depends on the second derivative; a second differentiation of eqn (47) yields

$$\frac{\partial^2 t^{(1)}}{\partial \lambda^2} = \frac{1}{6}\frac{\partial^4 \hat{W}}{\partial \lambda^4}(1, \lambda_3) - 2\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3) \quad \text{for} \quad \lambda = 1, \tag{56}$$

eqn (55) again having been used. For the strain-energy function eqn (51), eqn (56) simplifies to

$$\frac{\partial^2 t^{(1)}}{\partial \lambda^2} = \tfrac{1}{2}\mu(m^2 - 1)\lambda_3^{-(1/2)m} \quad \text{for} \quad \lambda = 1$$

and this reflects the intermediate role played by $|m| = 1$ in Figure 1.

The Taylor expansion

$$t^{(1)} = t_c^{(1)} + \tfrac{1}{2}(\lambda - 1)^2 \frac{\partial^2 t^{(1)}}{\partial \lambda^2} + \cdots$$

of eqn (47) near $\lambda = 1$ emphasizes that two deformation branches emanate from $(t_c^{(1)}, 1)$ for arbitrary $\hat{W}$ since, to the second order in $(\lambda - 1)$, it describes a parabola in the $(t^{(1)}, \lambda)$-plane.

Thus far, by fixing λ_3, we have restricted attention to the plane strain problem. The corresponding problem of all-round dead load with $t_1^{(1)}$, $t_2^{(1)}$, $t_3^{(1)}$ prescribed is also governed by eqn (45), and we conclude with some remarks about this. If, in particular, $t_1^{(1)} = t_2^{(1)}$ then, in the notation of eqns (35) and (36), eqn (45) becomes

$$(\lambda - \lambda^{-1})\lambda_3^{-1/2}t_1^{(1)} = \lambda \frac{\partial \hat{W}}{\partial \lambda}, \tag{57}$$

$$\lambda_3 t_3^{(1)} - \tfrac{1}{2}(\lambda + \lambda^{-1})\lambda_3^{-1/2}t_1^{(1)} = \lambda_3 \frac{\partial \hat{W}}{\partial \lambda_3}. \tag{58}$$

The results described in [10] for the neo-Hookean strain-energy function [$m = 2$ in (51)] are recovered by appropriate specialization of eqns (57) and (58), and generalized by examining the solutions of (57) and (58) for prescribed $t_1^{(1)}$ and $t_3^{(1)}$ along the lines described here for the plane-strain problem. We do not consider the general case here but illustrate the problem by taking $t_3^{(1)} = 0$ and using the strain-energy function (51). Elimination of $t_1^{(1)}$ from eqns (57) and (58) shows that either $\lambda = 1$ or λ_3 is given by

$$\lambda_3^{3m/2} = (\lambda^{m-1} - \lambda^{-m+1})/(\lambda^{-1} - \lambda), \tag{59}$$

which has value $1 - m$ in the limit $\lambda \to 1$. For eqn (59) to have a positive solution for λ_3 we must have $m < 1$. By taking $m = \tfrac{1}{2}$, for example, we obtain

$$\lambda_3 = (\lambda^{1/2} + \lambda^{-1/2})^{-4/3}, \qquad t_1^{(1)} = 2\mu\lambda_3,$$

which define an unstable path of deformation from the bifurcation point $t_1^{(1)} = \mu 2^{-1/3}$ on the path $\lambda = 1$.

Finally, on setting $t_3^{(1)} = t_1^{(1)}$ in eqn (58) we see that a possible solution of eqns (57) and (58) is $\lambda = \lambda_3 = 1$ for all $t_1^{(1)}$. A solution with $\lambda = 1$ and $\lambda_3 \neq 1$ is governed by

$$t_1^{(1)} = \lambda_3 \frac{\partial \hat{W}}{\partial \lambda_3}(1, \lambda_3)/(\lambda_3 - \lambda_3^{-1/2}) \tag{60}$$

and bifurcates from $\lambda = \lambda_3 = 1$ at the critical value

$$t_1^{(1)} = \frac{2}{3}\frac{\partial^2 \hat{W}}{\partial \lambda_3^2}(1, 1)$$

as $t_1^{(1)}$ increases. Secondary bifurcation into a deformation path with $\lambda \neq 1$ may occur subsequently when $t_1^{(1)}$ reaches the critical value obtained from eqn (57) in the limit $\lambda \to 1$, namely

$$t_1^{(1)} = \tfrac{1}{4}\lambda_3^{1/2}\frac{\partial^2 \hat{W}}{\partial \lambda^2}(1, \lambda_3),$$

where λ_3 and $t_1^{(1)}$ are also connected by eqn (60). In respect of the Mooney strain-energy function such bifurcations were analyzed in [1], but for eqn (51) the above critical values of $t_1^{(1)}$ are both equal to μ (independently of m).

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